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Results on a Transient Queue

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ABSTRACT

A result of P. Finch [Jour. Aus. Math. Soc., 10/61, 217-227] which obtains the generating function of the number of customers served during the busy period of $G/GI/1$ is extended. The case considered is that in which the service time of the first customer is a constant and service times of all subsequent customers are identically distributed random variables. The techniques employed involve a relation between the theories of fluctuation and of delayed recurrent events. A probabilistic derivation of a result due to Hirsch and Hanisch [Comm. Pure Appl. Math., 11/63] is given.

1. INTRODUCTION

P. Finch [4] has obtained an expression for the generating function of the number of customers served during the busy period of a general single-server queue assuming that the service time distribution of the first customer is the same as that of succeeding customers. In this paper we obtain a corresponding expression assuming that the service time distribution of the first customer is an arbitrary constant, (Theorem 1). As a by-product of this expression, we observe that results of H. Hanisch and W. Hirsch [5], previously obtained by analytical methods can be deduced by probabilistic arguments, (Theorem 3). Our method is based on a simple relation between the theories of fluctuation and of delayed recurrent events.

2. PRELIMINARY DEFINITIONS AND LEMMAS

Let X_1, X_2, \dots, X_n be an arbitrary sequence of mutually independent, identically distributed random variables defined on a probability space (Ω, \mathcal{S}, P) . Put $S_0 = 0$ and $S_n = \sum_{j=1}^n X_j$, $n \geq 1$.

Definitions:

- (1) $k \geq 1$ is a strict ladder index for (S_0, S_1, \dots, S_n) if $S_k > S_i$ for $i = 0, 1, \dots, k-1$.
- (2) $k \geq 1$ is the ρ -th strict ladder index if it is a

strict ladder index and is preceded by $\rho-1$ strict ladder indices.

Feller [1] has shown the result:

Lemma 1. The occurrence of a strict ladder index for $\{S_j\}_{j=0}^{\infty}$ is a recurrent event.

Let $A_n^{(\rho)}$ denote the event that n is the ρ -th strict ladder index of the sequence $\{S_j\}_{j=0}^{\infty}$. Let I be an interval in $[0, \infty]$ and let $\mu_n^{(\rho)}(I)$ denote the probability that $A_n^{(\rho)}$ and $S_n \in I$ occur. The following lemma, basic to our arguments, is an extension of a result of G. Szpizer due to G. Baxter [2].

Lemma 2. For $I \subset [0, \infty]$

$$(1) \quad \frac{1}{n} P\{S_n \in I\} = \sum_{\rho=1}^n \frac{1}{\rho} \mu_n^{(\rho)}(I) .$$

The following known results are also most pertinent.

Put $\mu_n^{(\rho)} = P\{A_n^{(\rho)}\}$, $n \geq 1$, and define $\mu_0^{(\rho)} = 0$. By Lemma 1, $\mu_n^{(\rho)}$ is the ρ -th convolution of $\mu_n^{(1)}$ with itself. In terms of generating functions of the sequence $\{\mu_n^{(\rho)}\}_{n=0}^{\infty}$,

$$(2) \quad \mu^{(\rho)}(t) = \sum_{n=1}^{\infty} \mu_n^{(\rho)} t^n = (\mu^{(1)}(t))^{\rho} .$$

Let I in (1) be $I_{\infty} = (0, \infty]$. Introducing generating functions in (1) we get

$$(3) \quad \sum_{n=1}^{\infty} \frac{1}{n} P\{S_n > 0\} t^n = \sum_{n=1}^{\infty} \sum_{\rho=1}^n \frac{1}{\rho} \mu_n^{(\rho)} t^n .$$

Note that $\mu_n^{(\rho)}(I_{\infty}) = \mu_n^{(\rho)}$. Since $\mu_n^{(\rho)} = 0$, $\rho > n$, the summation over ρ in (3) can be extended to ∞ . Interchanging the order of summation we get

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n} P\{S_n > 0\} t^n &= \sum_{\rho=1}^{\infty} \frac{1}{\rho} \sum_{n=1}^{\infty} \mu_n^{(\rho)} t^n \\ &= \sum_{\rho=1}^{\infty} \frac{1}{\rho} \mu^{(\rho)}(t) = \log \frac{1}{1 - \mu^{(1)}(t)} ; \end{aligned}$$

or

$$(4) \quad \mu^{(1)}(t) = 1 - e^{- \sum_{n=1}^{\infty} \frac{P\{S_n > 0\}}{n} t^n}$$

Now let $I_x = (0, x]$; set

$$\mu_n^{(\rho)}(I_x) = \tau_n^{(\rho)}(x) , \quad n \geq 1$$

$$\mu_0^{(\rho)}(I_x) = \tau_0^{(\rho)}(x) \equiv 0 .$$

Note that

$$(5) \quad \mu_n^{(\rho)} = \tau_n^{(\rho)}(\infty) , \quad n \geq 0 .$$

Now n is the ρ -th strict ladder index if and only if some $k \leq n$ is the first of ρ successive strict ladder indices and n is the ρ -th. Furthermore if $S_k = y$, $0 \leq y \leq x$, then by Lemma 1

$$P\{A_n^{(\rho)} \cap S_n \in I_x / A_k^{(1)} \cap S_k = y\} = \tau_{n-k}^{(\rho-1)}(x-y) .$$

Since $0 \leq y \leq x$ and $k = 1, 2, \dots, n-1$

$$(6) \quad \tau_n^{(\rho)}(x) = \sum_{k=1}^{n-1} \int_0^x \tau_{n-k}^{(\rho-1)}(x-y) d\tau_k^{(1)}(y), \quad n \geq 2, \quad \rho \geq 2.$$

$\tau_n^{(\rho)}(x)$ is nondecreasing in x . Thus (6) is well-defined as a Lebesgue-Stieltjes integral.

We define also

$$\tilde{\tau}_n^{(\rho)}(s) = \int_0^\infty e^{-sx} d\tau_n^{(\rho)}(x) , \quad s > 0 ,$$

the Laplace-Stieltjes transform of $\tau_n^{(\rho)}(x)$. Let

$$\tau^{(\rho)}(t, s) = \sum_{n=1}^\infty \tilde{\tau}_n^{(\rho)}(s) t^n .$$

From (6) we obtain

$$(7) \quad \tilde{\tau}^{(\rho)}(t, s) = (\tilde{\tau}^{(1)}(t, s))^{\rho} ,$$

which is valid for $\rho \geq 1$.

We obtain next an expression for $\tilde{\tau}^{(1)}(t, s)$ in terms of the transform,

$$\tilde{b}_n(s) = \int_0^\infty e^{-sx} db_n(x)$$

of the non-decreasing function

$$b_n(x) = P\{0 < S_n \leq x\} .$$

From Lemma 2 we have

$$\frac{1}{n} b_n(x) = \sum_{\rho=1}^n \frac{1}{\rho} \tilde{\tau}_n^{(\rho)}(x)$$

which, in terms of Stieltjes transforms, becomes

$$\frac{1}{n} \tilde{b}_n(s) = \sum_{\rho=1}^n \frac{1}{\rho} \tilde{\tau}_n^{(\rho)}(s) .$$

Forming generating functions and observing that

$$\tilde{\tau}_n^{(\rho)}(s) \equiv 0 , \quad \rho > n$$

we have after interchanging the order of summation

$$\sum_{n=1}^{\infty} \frac{1}{n} \tilde{b}_n(s) t^n = \sum_{\rho=1}^{\infty} \frac{1}{\rho} (\tilde{\tau}^{(1)}(t, s))^{\rho} = \log \frac{1}{1 - \tilde{\tau}^{(1)}(t, s)} ;$$

thus

$$(8) \quad \tilde{\tau}^{(1)}(t, s) = 1 - e^{- \sum_{n=1}^{\infty} [\tilde{b}_n(s)/n] t^n}$$

The problem treated in this paper is concerned with sequences of the form Y, X_1, X_2, \dots where Y is independent of the sequence $\{X_j\}_{j=1}^{\infty}$, the random variables X_j , $j = 1, 2, \dots$ are independent and identically distributed, and Y has an arbitrary distribution.

Define the partial sums

$$(9) \quad \sigma_0 = 0, \quad \sigma_1 = Y, \dots \quad \sigma_k = Y + \sum_{i=1}^{k-1} X_i, \dots$$

Let E denote the event the occurrence of a strict ladder index in the sequence $\{\sigma_j\}_{j=0}^{\infty}$. We observe from Lemma 2, that after the first occurrence of E , E behaves as a recurrent event; hence we have:

Lemma 3. The occurrence of a strict ladder index for the $\{\sigma_j\}_{j=0}^{\infty}$ is a delayed recurrent event.

Let

$$\phi_n^{(\rho)}(-Y) = P\{n \text{ is the } \rho\text{-th strict ladder index among } \{\sigma_j\}_{j=0}^{\infty}\}, \quad n \geq 1, \rho \geq 1.$$

From Lemma 3 we conclude that

$$(10) \quad \{\phi_n^{(\rho+1)}(-Y)\} = \{\phi_n^{(1)}(-Y)\} * \{\mu_n^{(\rho)}\}, \quad n \geq 1, \rho \geq 1,$$

where $*$ denotes convolution. Define $\phi_0^{(\rho)}(-Y) = 0$ and put

$$\phi^{(\rho)}(t, -Y) = \sum_{n=1}^{\infty} \phi_n^{(\rho)}(-Y) t^n.$$

In terms of generating functions, (10) becomes

$$\phi^{(\rho)}(t, -Y) = \phi^{(1)}(t, -Y) (\mu^{(1)}(t))^{\rho-1}, \quad \rho \geq 1.$$

For the sequence Y, X_1, X_2, \dots let

$$(11) \quad S_0 = 0, \quad S_j = \sum_{i=1}^j X_i, \quad j \geq 1.$$

The following lemma relates strict ladder indices

of $\{\sigma_j\}_{j=0}^{\infty}$ to those of $\{S_j\}_{j=0}^{\infty}$.

Lemma 4. The sequence $\{\sigma_j\}_{j=0}^{\infty}$ has n as a strict ladder index if and only if the sequence $\{S_j\}_{j=0}^{\infty}$ has $n-1$ as a strict ladder index and $S_{n-1} > Y$, $n \geq 2$.

Proof. If n is a strict ladder index of $\{\sigma_j\}_{j=0}^{\infty}$, we have

$$(12) \quad \sigma_n > \sigma_{n-k}, \quad k = 1, 2, \dots, n.$$

Since

$$\sigma_j = Y + S_{j-1}, \quad j = 1, 2, \dots, n,$$

(12) can be written in the form

$$S_{n-1} > S_{n-k-1}, \quad k = 1, 2, \dots, n$$

where we define

$$S_{-1} = -Y.$$

Thus $S_{n-1} > -Y$, and $n-1$ is a strict ladder index for $\{S_j\}_{j=0}^{\infty}$. The argument is clearly invertible.

3. THE SINGLE-SERVER QUEUE

We turn our attention now to a single-server queueing system in which customers arrive at the instants $t_1, t_2, \dots, t_n, \dots$ and the inter-arrival times $\tau_n = t_{n+1} - t_n$, $n \geq 1$, $t_1 = 0$ are independent, identically distributed, non-negative random variables with distribution function F . We assume that the first customer to arrive finds the server idle, and service starts immediately. The service time of this first customer is a fixed quantity b . Successive customers are serviced in any order, and the server is never idle if customers are waiting for service. The service times of customers after the first form a sequence $\{\delta_n\}_{n=2}^{\infty}$ of identically distributed non-negative random variables independent of each other and of the input process $\{t_n\}_{n=1}^{\infty}$, with common distribution function G . Both F and G are taken to be continuous from the right.

By the busy period we mean the time interval from t_1 , until the server is next idle.

There is a duality between the behavior of a single server queue during a busy period and the following moving queue problem. Let $\{E_j\}$, $j = 1, 2, \dots$ denote the members of an ordered infinite queue, the spacing between adjacent elements being stochastically determined. Let b denote the distance of E_1 at time $t = 0$ from a barrier located at the origin. Let s_j denote the distance between E_j and E_{j+1} . We assume that at time $t = 0$ service begins on E_1 and the queue begins to move with unit velocity towards the barrier. Service of E_j is initiated instantaneously upon completion of service on E_{j-1} . Let τ_j denote the service time on the j -th item. If an element E_j arrives at 0 before service on it has been completed, we assume the process to terminate. If we identify the distribution functions of $\{s_j\}$ and $\{\tau_j\}$ with those of the general single server queue, then the number of elements serviced before an item reaches the barrier is equal to one less than the number of elements serviced in a busy period.

The process described has a recursive character, which is best seen by arguing in terms of the moving server problem. Let v denote the number of elements served before termination of the process. Set

$$P\{v = n / E_1 \text{ at } b\} = \hat{p}_n(b) .$$

Since $v = 0$ if and only if the service time on E_1 exceeds b , we have

$$\hat{p}_0(b) = 1 - F(b) .$$

Moreover, v takes the value $n \geq 1$ if and only if service on E_1 is completed and exactly $n-1$ additional elements are serviced prior to termination; denoting this event by A we have,

$$P\{A / \tau_1 = t, \mathcal{S}_1 = x\} = \hat{p}_{n-1}(b - t + x) .$$

Hence,

$$\hat{p}_n(b) = \int_{[0, b]} \int_{[0, \infty)} \hat{p}_{n-1}(b - t + x) dF(t) dG(x) , \quad n \geq 1 .$$

Let $p_{n+1}(b)$ be the corresponding busy period probability. We have

$$p_{n+1}(b) = \hat{p}_n(b) , \quad n \geq 0 .$$

Let

$$Y = \tau_1 - b$$

$$X_{j-1} = \tau_j - \mathcal{S}_j , \quad j \geq 2 .$$

Let $u \leq b$ be a realization of τ_1 , and let $\{S_j\}_{j=0}^{\infty}$

be defined as in (9) and (11) for the sequence of random variables

$$u - b, x_1, x_2, \dots$$

Let

$$\phi_n^{(\rho)}(b-u) = P\{n \text{ is the } \rho\text{-th strict ladder index among } \{\sigma_j\}_{j=0}^{\infty}\}$$

$$\lambda_n^{(\rho)}(b-u) = P\{n \text{ is the } \rho\text{-th strict ladder index among } \{s_j\}_{j=0}^{\infty} \text{ and } s_n > b-u\},$$

$$\tau_n^{(\rho)}(b-u) = P\{n \text{ is the } \rho\text{-th strict ladder index among } \{s_j\}_{j=0}^{\infty} \text{ and } 0 < s_n \leq b-u\},$$

$$\mu_n^{(\rho)} = P\{n \text{ is the } \rho\text{-th strict ladder index among } \{s_j\}_{j=0}^{\infty}\}, \quad n \geq 1, \quad \rho = 1, 2, \dots, n;$$

all functions vanish at $n = 0$. The following relation is trivially evident.

$$(14) \quad \mu_{n-1}^{(\rho)} = \tau_{n-1}^{(\rho)}(b-u) + \lambda_{n-1}^{(\rho)}(b-u), \quad n \geq 1.$$

Lemma 5.

$$\sum_{\rho=1}^n \phi_n^{(\rho)}(b-u) = \sum_{\rho=1}^{n-1} \lambda_{n-1}^{(\rho)}(b-u), \quad n \geq 1.$$

Proof. Let

$$A_n = \{n \text{ is a strict ladder index for } \{\sigma_j\}_{j=0}^{\infty}\} ,$$

$$B_n = \{n \text{ is a strict ladder index for } \{s_j\}_{j=0}^{\infty} \text{ and } s_n > b - u\}, \quad n \geq 1.$$

Clearly,

$$P\{A_n\} = \sum_{\rho=1}^{n-1} \phi_n^{(\rho)}(b-u), \quad n \geq 1,$$

and

$$P\{B_{n-1}\} = \sum_{\rho=1}^{n-1} \lambda_{n-1}^{(\rho)}(b-u), \quad n \geq 2.$$

Moreover, by Lemma 4, $A_n = B_{n-1}$, which completes the proof for $n \geq 2$. For $n = 1$ we note that for $u \leq b$,

$$\phi_1^{(1)}(b-u) = 0.$$

Since the busy period ends if the cumulative service time is less than the cumulative inter-arrival times, we have

$$(15) \quad p_n(b) = \int_{[0,b]} \phi_n^{(1)}(b-u) dF(u), \quad n \geq 2.$$

It is convenient to write this equation in the form

$$(16) \quad p_n(b) = \int_{[0,b]} (\phi_n^{(1)}(b-u) - \phi_n^{(1)}(0)) dF(u) + F(b) \phi_n^{(1)}(0).$$

Lemma 6. $p_n(b)$ and $\phi_n(b)$ are of bounded variation.

Proof. Let

$$(17) \quad \pi_n(b) = P\{\text{servicing at least } n \text{ customers}\}.$$

Then

$$p_n(b) = \pi_n(b) - \pi_{n+1}(b).$$

H. Hanisch and W. Hirsch [5] have shown that $\pi_n(b)$ is non-decreasing in b . Hence $p_n(b)$ is of bounded variation.

To establish that $\phi_n(b)$ is of bounded variation, we observe from (13) and (15) that

$$\phi_n(b) = \int_{[0, \infty)} p_{n-1}(b+x) dG(x), \quad b \geq 0, \quad n \geq 2.$$

Since p_n is of bounded variation, so is ϕ_n .

Forming Laplace-Stieltjes transforms of (16) we get

$$(18) \quad \tilde{p}_n(s) = \tilde{\phi}_n^{(1)}(s) \tilde{F}(s) + \tilde{\phi}_n^{(1)}(0) \tilde{F}(s), \quad n \geq 2,$$

(where " \sim " denotes transform). Now putting

$$(19) \quad \tilde{p}(t, s) = \sum_{n=0}^{\infty} \tilde{p}_n(s) t^n,$$

and denoting by $F^{(n)}$ and $G^{(n)}$ the n -th convolutions of F

and G respectively, we have

Theorem 1.

$$(20) \quad \tilde{P}(t, s) = t(\tilde{p}_1(s) + \tilde{F}(s) \tilde{\phi}^*(t, s)) ,$$

where

$$(21) \quad \tilde{\phi}^*(t, s) = \sum_{j=1}^{\infty} c_j t^j ;$$

$$(22) \quad c_j =$$

$$\sum_{k_1+2k_2+\dots+jk_j=j} \frac{(-1)^{k_1+k_2+\dots+k_j+1}}{k_1!k_2!\dots k_j!} \left(\frac{a_1-\tilde{b}_1(s)}{1}\right)^{k_1} \dots \left(\frac{a_j-\tilde{b}_j(s)}{j}\right)^{k_j} ;$$

$$(23) \quad a_n = \int_{[0, \infty)} (1 - F^{(n)}(t)) dG^{(n)}(t) ,$$

and $\tilde{b}_n(s)$ is the Laplace-Stieltjes transform of

$$(24) \quad b_n(x) = \int_{[0, \infty)} (F^{(n)}(x+t) - F^{(n)}(t)) dG^{(n)}(t) , \quad n \geq 1 .$$

Proof. Let

$$\tilde{\phi}^{(1)}(t, s) = \sum_{n=0}^{\infty} \tilde{\phi}_n^{(1)}(s) t^n$$

and note that

$$\begin{aligned}\phi_0^{(1)}(u) &= \phi_1^{(1)}(u) = 0, & u \geq 0 \\ \phi_n^{(1)}(0) &= \mu_{n-1}^{(1)}, & n \geq 1.\end{aligned}$$

Introducing generating functions in (18), we get

$$(25) \quad \tilde{P}(t, s) = t \tilde{p}_1(s) + \tilde{\phi}^{(1)}(t, s) \tilde{F}(s) + t \tilde{\mu}^{(1)}(t) \tilde{F}(s).$$

By Lemma 5 and (14), we have

$$(26) \quad \sum_{\rho=1}^n \phi_n^{(\rho)}(b-u) = \sum_{\rho=1}^{n-1} \mu_{n-1}^{(\rho)} - \sum_{\rho=1}^{n-1} \tau_{n-1}^{(\rho)}(b-u), \quad n \geq 2, \quad u \leq b.$$

We introduce Laplace-Stieltjes transforms in (26) and get

$$(27) \quad \sum_{\rho=1}^{\infty} \phi_n^{(\rho)}(s) = \sum_{\rho=1}^{\infty} \tau_{n-1}^{(\rho)}(s), \quad n \geq 2$$

where we have extended the upper limits of the summations to ∞ since $\phi_n^{(\rho)}(u) = 0$, $\rho > n$, and $\tau_{n-1}^{(\rho)}(u) = 0$, $\rho > n-1$.

Introducing generating functions in (27), we have after interchanging the order of summation

$$\sum_{\rho=1}^{\infty} \tilde{\phi}^{(\rho)}(t, s) = -t \sum_{\rho=1}^{\infty} \tilde{\tau}^{(\rho)}(t, s).$$

Referring back to the basic results from fluctuation theory

(2) and (7), we get

$$\frac{\tilde{\phi}^{(1)}(t,s)}{1-\mu^{(1)}(t)} = \frac{-t \tilde{\tau}^{(1)}(t,s)}{1-\tilde{\tau}^{(1)}(t,s)}$$

which when substituted into (25) yields

$$(28) \quad \tilde{P}(t,s) = t \tilde{p}_1(s) + t \tilde{F}(s) \tilde{\phi}^*(t,s)$$

where

$$\tilde{\phi}^*(t,s) = 1 - \frac{1-\mu^{(1)}(t)}{1-\tilde{\tau}^{(1)}(t,s)}.$$

From our earlier results (5) and (8) we see that

$$(29) \quad \tilde{\phi}^*(t,s) = 1 - \frac{e^{-\sum_{n=1}^{\infty} \frac{P(S_n > 0)}{n} t^n}}{e^{-\sum_{n=1}^{\infty} \frac{b_n(s)}{n} t^n}}, \quad t < 1.$$

Let $a_n = P\{S_n > 0\}$. a_n and $b_n(x)$ can be expressed in terms of distributions of the process as given by (21) and (22). Let

$$1 - e^{-\sum_{n=1}^{\infty} \frac{a_n - b_n(s)}{n} t^n} = \sum_{j=1}^{\infty} c_j t^j, \quad t < 1.$$

By a combinatorial result (Cf. Riordan [7]), c_j is given by (20). Thus from (28)

$$\tilde{p}_{j+1}(s) = c_j \tilde{F}(s), \quad j = 1, 2, \dots$$

and we have the initial condition that

$$\tilde{p}_1(s) = \int_0^{\infty} e^{-sb} d(1 - F(b)).$$

4. PROPERTIES OF $P(l, b)$

From our knowledge of the generating function $P(t, s)$ we can discuss properties of $P(l, b) = P\{\text{finitely many customers are serviced in a busy period}\}$ or in terms of the dual formulation the $P\{\text{finitely many elements are serviced before absorption}\}$. This quantity can be regarded as one measure of the effectiveness of the service facility.

P. Finch has shown the following result for the queueing system $G|GI|1$ where τ is the inter-arrival time and \mathcal{S} is the service time.

Theorem 2. If $E(\tau - \mathcal{S}) < \infty$, then $\mu^{(1)}(1) = 1$ if $E(\tau - \mathcal{S}) \geq 0$ and F and G do not both degenerate at the same point; and $\mu^{(1)}(1) < 1$ if $E(\tau - \mathcal{S}) < 0$.

We can extend Finch's theorem to include the following cases of infinite expectations.

Theorem 2'. (a) If $\infty = E(\tau - \mathcal{S})$, $\infty = E(\tau) > E(\mathcal{S})$, then $\mu^{(1)}(1) = 1$; and if (b) $-\infty = E(\tau - \mathcal{S})$, $E(\mathcal{S}) = \infty > E(\tau)$, then $\mu^{(1)}(1) < 1$.

Proof. Suppose (a) is the case. Define the truncated variables

$$\begin{aligned} x_n^c &= x_n & x_n < c \\ &= 0 & x_n > c \end{aligned}$$

where $0 < c < \infty$ is chosen so that $0 < E(x_n^c) < \infty$. Let

$$S_n^c = \sum_{j=0}^n x_j^c, \quad A_n^c = P(S_n^c > 0).$$

We have the relation between events

$$(30) \quad \{S_n^c > 0\} \subseteq \{S_n > 0\}.$$

Let

$$\mu_c^{(1)}(t) = 1 - e^{-\sum_{n=1}^{\infty} \frac{A_n^c}{n} t^n}$$

Since $\infty > E(x_n^c) > 0$, we obtain from Theorem 2, $\mu_c^{(1)}(1) = 1$ which implies $\sum_{n=1}^{\infty} A_n^c/n = \infty$. Now by (30), $A_n^c \leq A_n$ and, therefore, $\sum_{n=1}^{\infty} A_n/n = \infty$. From (4) we then obtain $\mu^{(1)}(1) = 1$.

We similarly consider the case $E(\tau - \delta) = -\infty$.

We now define the truncated variables

$$x_n^c = x_n, \quad x_n > -c$$

$$= 0, \quad x_n \leq -c$$

where $0 < c < \infty$ is chosen so that $0 > E(x_n^c) > \infty$.

Let s_n^c and a_n^c be defined accordingly, and we now observe

$$(31) \quad \{s_n > 0\} \subseteq \{s_n^c > 0\}$$

and hence $a_n \leq a_n^c$. $\mu_c^{(1)}(1) < 1$ follows from Theorem 2. Therefore $\sum a_n^c/n < \infty$ which implies $\sum a_n/n < \infty$ and hence $\mu^{(1)}(1) < 1$.

The following lemmas are required for the proof of the main theorem of this section.

Lemma 7.

$$\tilde{P}(l, s) = \int_0^\infty e^{-sb} dP(l, b).$$

Proof. We have defined

$$(32) \quad \tilde{P}(l, s) = \sum_{n=0}^{\infty} \int_0^\infty e^{-sb} dp_n(b)$$

which can alternatively be written as

$$\lim_{N \rightarrow \infty} \sum_{n=0}^N \int_0^\infty e^{-sb} dp_n(b) = \lim_{N \rightarrow \infty} \int_0^\infty e^{-sb} \sum_{n=0}^N dp_n(b),$$

the interchange being permitted over the finite interval.

Noting that as defined by (17)

$$\sum_{n=0}^N p_n(b) = 1 - \pi_{N+1}(b) ,$$

we have from (32)

$$(33) \quad \tilde{P}(l, s) = - \lim_{N \rightarrow \infty} \int_0^{\infty} e^{-sb} d\pi_{N+1}(b) .$$

H. Hanisch and W. Hirsch in [5] have shown $\pi_N(b)$ to be a distribution function. Now let $S_n = \{\text{at least } n \text{ customers are serviced}\}$, $n \geq 1$, and

$$S = \{\text{infinitely many customers are serviced}\} .$$

Clearly

$$S_{n+1} \subset S_n , \quad \bigcap_{n=1}^{\infty} S_n = S .$$

Thus

$$\lim_{n \rightarrow \infty} P(S_n) = P(S)$$

i.e.,

$$\lim_{n \rightarrow \infty} \pi_n(b) = 1 - P(l, b) .$$

Thus the Helly-Bray Theorem applied to (33) yields

$$(34) \quad \begin{aligned} \tilde{P}(l, s) &= - \lim_{N \rightarrow \infty} \int_0^{\infty} e^{-sb} d\pi_{N+1}(b) \\ &= \int_0^{\infty} e^{-sb} dP(l, b) . \end{aligned}$$

The inequality $\tilde{P}(l, s) \leq 1$ follows from (34) since

$$\int_0^{\infty} e^{-sb} dP(l, b) \leq P(l, \infty) \leq 1.$$

Lemma 8.

$$\tilde{\tau}^{(1)}(t, s) = \int_0^{\infty} e^{-sb} d\tau^{(1)}(t, b).$$

Proof. We have defined

$$\begin{aligned} \tilde{\tau}^{(1)}(t, s) &= \sum_{n=0}^{\infty} \int_0^{\infty} e^{-sb} d\tau_n^{(1)}(b) t^n \\ &= \lim_{N \rightarrow \infty} \int_0^{\infty} e^{-sb} d \sum_{n=0}^N \tau_n^{(1)}(b) t^n . \end{aligned}$$

Now $\sum_{n=0}^N \tau_n^{(1)}(b) t^n$ is monotone non-decreasing and right continuous in b , inheriting these properties from $\{\tau_n^{(1)}(b)\}$. Furthermore, using these facts and (5), we deduce

$$\sum_{n=0}^N \tau_n^{(1)}(b) t^n \leq \sum_{n=0}^{\infty} \tau_n^{(1)}(\infty) = \mu^{(1)}(1) \leq 1,$$

and thus

$$\left\{ \sum_{n=0}^N \tau_n^{(1)}(b) t^n \right\}_{N=0}^{\infty}$$

is a set of distribution functions.

Thus, by the Helly-Bray Theorem,

$$\lim_{N \rightarrow \infty} \int e^{-sb} d \sum_{n=0}^N \tau_n^{(1)}(b) t^n = \int_0^{\infty} e^{-sb} d \tau^{(1)}(t, b).$$

Now we can prove the following.

Theorem 3. If (a) $E(X) = E(\tau - \delta) \geq 0$, and
 F and G do not both degenerate at the same point, then
 $P(1, b) \equiv 1$; if (b) $F(\infty) = 1$, $E(\tau - \delta) < 0$, then

$$\lim_{b \rightarrow \infty} P(1, b) = 0.$$

Proof. An application of Abel's theorem to (20) yields

$$(35) \quad \tilde{P}(1, s) = \lim_{t \rightarrow 1} t(\tilde{p}_1(s) + \tilde{F}(s) \tilde{\phi}^*(t, s)).$$

We have from Theorem 2 that under (a), $\mu^{(1)}(1) = 1$.

hence

$$\lim_{t \rightarrow 1} \tilde{\phi}^*(t, s) = \lim_{t \rightarrow 1} \frac{\mu^{(1)}(t) - \tilde{\tau}^{(1)}(t, s)}{1 - \tilde{\tau}^{(1)}(t, s)} = 1.$$

From (35) we then have

$$(36) \quad \tilde{P}(1, s) = \tilde{p}_1(s) + \tilde{F}(s)$$

which by Lemma 7, when inverted, yields

$$P(1, b) = 1 - F(b) + F(b) = 1.$$

Under (b) we have the result $\mu^{(1)}(1) < 1$.

From Lemma 8 we conclude

$$\begin{aligned} \tilde{\tau}^{(1)}(t, s) &= \int_0^\infty e^{-sb} d\tau^{(1)}(t, b) \\ &\leq \tau^{(1)}(t, \infty) = \mu^{(1)}(t) < 1. \end{aligned}$$

Hence

$$(37) \quad \lim_{t \rightarrow 1} \tilde{\phi}^*(t, s) = \frac{\mu^{(1)}(1) - \tilde{\tau}^{(1)}(1, s)}{1 - \tilde{\tau}^{(1)}(1, s)} < \infty.$$

Using the Abelian convergence theorem for Laplace transforms we have

$$\lim_{s \rightarrow 0} \tilde{\tau}(1, s) = \tau^{(1)}(1, \infty) = \mu^{(1)}(1),$$

and thus from (37), $\lim_{s \rightarrow 0} \phi^*(1, s) = 0$.

From (35) we then conclude $\lim_{s \rightarrow 0} \tilde{p}(1, s) = \lim_{s \rightarrow 0} \tilde{p}_1(s)$.

Again utilizing the Abelian convergence theorem

and the fact that $F(\infty) = 1$, we get

$$\lim_{s \rightarrow 0} \tilde{p}(1, s) = \lim_{b \rightarrow \infty} P(1, b) - P(1, 0^-) = \lim_{s \rightarrow 0} \tilde{p}_1(s) = -1.$$

Now $P(1, 0^-) = 1$. Hence $\lim_{b \rightarrow \infty} P(1, b) = 0$.

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